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Explicit results for all orders of the ε-expansion of certain massive and massless diagrams

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Abstract

An arbitrary term of the ε-expansion of dimensionally regulated off-shell massless one-loop three-point Feynman diagram is expressed in terms of log-sine integrals related to the polylogarithms. Using magic connection between these diagrams and two-loop massive vacuum diagrams, the ε-expansion of the latter is also obtained, for arbitrary values of the masses. The problem of analytic continuation is also discussed.

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1. In this paper we shall discuss some issues related to the evaluation of Feynman integrals in the framework of dimensional regularization [1], when the space-time dimension is $n = 4 - 2\varepsilon$. Sometimes it is possible to present results valid for an arbitrary ε, usually in terms of hypergeometric functions. However, for practical purposes the coefficients of the expansion in ε are important. In particular, in multi-loop calculations higher terms of the ε-expansion of one- and two-loop functions are needed, since one can get contributions where these functions are multiplied by singular factors containing poles in ε. Such poles may appear not only due to factorizable loops, but also as a result of applying the integration by parts [2] or other techniques [3].

For the one-loop two-point function $J^{(2)}(n;\nu_1,\nu_2)$ with external momentum k_{12}, masses m_1 and m_2 and unit powers of propagators $\nu_1 = \nu_2 = 1$, we have obtained the following result for an arbitrary term of the ε-expansion (see ref. [4]):

$$J^{(2)}(4-2\varepsilon;1,1) = \mathrm{i}\pi^{2-\varepsilon}\frac{\Gamma(1+\varepsilon)}{2(1-2\varepsilon)}\left\{\frac{m_1^{-2\varepsilon}+m_2^{-2\varepsilon}}{\varepsilon} + \frac{m_1^2-m_2^2}{\varepsilon\, k_{12}^2}\left(m_1^{-2\varepsilon}-m_2^{-2\varepsilon}\right)\right.$$
$$\left. - \left[\Delta(m_1^2,m_2^2,k_{12}^2)\right]^{1/2-\varepsilon}\sum_{j=0}^{\infty}\frac{(2\varepsilon)^j}{j!}\left[\mathrm{Ls}_{j+1}(2\tau'_{01})+\mathrm{Ls}_{j+1}(2\tau'_{02})-2\mathrm{Ls}_{j+1}(\pi)\right]\right\}, \quad (1)$$

where

$$\cos\tau_{12} = \frac{m_1^2+m_2^2-k_{12}^2}{2m_1m_2}, \qquad \cos\tau'_{01} = \frac{m_1^2-m_2^2+k_{12}^2}{2m_1\sqrt{k_{12}^2}}, \qquad \cos\tau'_{02} = \frac{m_2^2-m_1^2+k_{12}^2}{2m_2\sqrt{k_{12}^2}}. \quad (2)$$

The angles τ'_{0i} $(i=1,2)$ are related to the angles τ_{0i} used in refs. [4,5] via $\tau'_{0i} = \pi/2 - \tau_{0i}$. The "triangle" function Δ is defined as

$$\Delta(x,y,z) = 2xy + 2yz + 2zx - x^2 - y^2 - z^2 = -\lambda(x,y,z), \quad (3)$$

where $\lambda(x,y,z)$ is the Källen function. The coefficient of ε^j has a closed form in terms of log-sine integrals (see in [6], chapter 7.9),

$$\mathrm{Ls}_j(\theta) \equiv -\int_0^\theta \mathrm{d}\theta' \ln^{j-1}\left|2\sin\frac{\theta'}{2}\right|. \quad (4)$$

In particular, $\mathrm{Ls}_1(\theta) = -\theta$ and $\mathrm{Ls}_2(\theta) = \mathrm{Cl}_2(\theta)$, where

$$\mathrm{Cl}_j(\theta) = \begin{cases} \mathrm{Im}\left[\mathrm{Li}_j\left(e^{\mathrm{i}\theta}\right)\right] = \left[\mathrm{Li}_j\left(e^{\mathrm{i}\theta}\right) - \mathrm{Li}_j\left(e^{-\mathrm{i}\theta}\right)\right]/(2\mathrm{i}), & j \text{ even} \\ \mathrm{Re}\left[\mathrm{Li}_j\left(e^{\mathrm{i}\theta}\right)\right] = \left[\mathrm{Li}_j\left(e^{\mathrm{i}\theta}\right) + \mathrm{Li}_j\left(e^{-\mathrm{i}\theta}\right)\right]/2, & j \text{ odd} \end{cases} \quad (5)$$

is the Clausen function (see in [6]), Li_j is the polylogarithm. Note that the values of $\mathrm{Ls}_j(\pi)$ can be expressed in terms of Riemann's ζ function, see eqs. (7.112)–(7.113) of [6]. Moreover, the infinite sum with $\mathrm{Ls}_j(\pi)$ in (2) can be converted into Γ functions (see eq. (16) below).

The arguments of Ls functions in eq. (2) have simple geometrical interpretation [5]. We note that $\tau_{12}+\tau'_{01}+\tau'_{02}=\pi$ (equivalently, $\tau_{01}+\tau_{02}=\tau_{12}$). Therefore, τ_{12}, τ'_{01} and τ'_{02} can be understood as the angles of a triangle whose sides are m_1, m_2 and $\sqrt{k_{12}^2}$, whereas the area of this triangle is $\frac{1}{4}\sqrt{\Delta(m_1^2,m_2^2,k_{12}^2)}$. The ε-expansion (2) is directly applicable in the region where $\Delta(m_1^2,m_2^2,k_{12}^2)\geq 0$, i.e. when $(m_1-m_2)^2\leq k_{12}^2\leq(m_1+m_2)^2$. In other regions, the proper analytic continuation should be constructed. We note that the result for the ε-term was obtained in [7]. For the case $m_1=0$ ($m_2\equiv m$), the first terms of the expansion (up to ε^3) are presented in eq. (A.3) of [8].

We shall show that similar explicit results can be constructed for the off-shell massless one-loop three-point function with external momenta p_1, p_2 and p_3 ($p_1+p_2+p_3=0$),

$$J(n;\ \nu_1,\nu_2,\nu_3|p_1^2,p_2^2,p_3^2)\equiv\int\frac{\mathrm{d}^n r}{[(p_2-r)^2]^{\nu_1}\,[(p_1+r)^2]^{\nu_2}\,(r^2)^{\nu_3}}, \tag{6}$$

as well as for the two-loop vacuum diagram with arbitrary masses m_1, m_2 and m_3,

$$I(n;\ \nu_1,\nu_2,\nu_3|m_1^2,m_2^2,m_3^2)\equiv\int\int\frac{\mathrm{d}^n p\ \mathrm{d}^n q}{(p^2-m_1^2)^{\nu_1}\,(q^2-m_2^2)^{\nu_2}\,[(p-q)^2-m_3^2]^{\nu_3}}. \tag{7}$$

According to the magic connection [9], they are closely related to each other. To be specific, we shall consider the case $\nu_1=\nu_2=\nu_3=1$, although the approach can be also applied to arbitrary integer values of ν_i.

2. Thus, our purpose is to obtain an arbitrary term of the ε-expansion of dimensionally regulated integrals $J(4-2\varepsilon;1,1,1)$ and $I(4-2\varepsilon;1,1,1)$ defined in eqs. (6) and (7), respectively. As in [9], we assume that $p_i^2\leftrightarrow m_i^2$. Moreover, below we shall omit the arguments p_i^2 and m_i^2 in the integrals J and I, respectively. We shall mainly be interested in the region where all p_i^2 are positive (time-like), whereas

$$\Delta(p_1^2,p_2^2,p_3^2)\equiv-\lambda(p_1^2,p_2^2,p_3^2)\geq 0. \tag{8}$$

According to the magic connection (we use eq. (16) of [9], with changed sign of ε),

$$J(4-2\varepsilon;1,1,1)=\pi^{-3\varepsilon}\,\mathrm{i}^{1+2\varepsilon}\,\left(p_1^2p_2^2p_3^2\right)^{-\varepsilon}\,\frac{\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\,I(2+2\varepsilon;1,1,1). \tag{9}$$

Then, for $I(2+2\varepsilon;1,1,1)$ we can use an exact result (4.12)–(4.13) from [10] (see also in [11]), in terms of hypergeometric ${}_2F_1$ functions. Using the well-known transformation

$${}_2F_1\left(\left.\begin{array}{c}\varepsilon,1\\ 1/2+\varepsilon\end{array}\right|z\right)=\frac{1}{1-z}\ {}_2F_1\left(\left.\begin{array}{c}1,1/2\\ 1/2+\varepsilon\end{array}\right|\frac{z}{z-1}\right), \tag{10}$$

and then changing $\varepsilon\to 1-\varepsilon$, we arrive at

$$J(4-2\varepsilon;1,1,1)=2\pi^{2-\varepsilon}\,\mathrm{i}^{1+2\varepsilon}\,\left(p_1^2p_2^2p_3^2\right)^{-\varepsilon}\,\frac{\Gamma^2(1-\varepsilon)\,\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}$$

$$
\times \left\{ \frac{(p_1^2 p_2^2)^\varepsilon}{p_1^2+p_2^2-p_3^2} \; {}_2F_1\left(\begin{array}{c} 1, 1/2 \\ 3/2-\varepsilon \end{array} \middle| - \frac{\Delta(p_1^2,p_2^2,p_3^2)}{(p_1^2+p_2^2-p_3^2)^2} \right) \right.
$$

$$
+ \frac{(p_2^2 p_3^2)^\varepsilon}{p_2^2+p_3^2-p_1^2} \; {}_2F_1\left(\begin{array}{c} 1, 1/2 \\ 3/2-\varepsilon \end{array} \middle| - \frac{\Delta(p_1^2,p_2^2,p_3^2)}{(p_2^2+p_3^2-p_1^2)^2} \right)
$$

$$
+ \frac{(p_3^2 p_1^2)^\varepsilon}{p_3^2+p_1^2-p_2^2} \; {}_2F_1\left(\begin{array}{c} 1, 1/2 \\ 3/2-\varepsilon \end{array} \middle| - \frac{\Delta(p_1^2,p_2^2,p_3^2)}{(p_3^2+p_1^2-p_2^2)^2} \right)
$$

$$
\left. - \pi \frac{\Gamma(2-2\varepsilon)}{\Gamma^2(1-\varepsilon)} \; \left[\Delta(p_1^2,p_2^2,p_3^2) \right]^{-1/2+\varepsilon} \; \Theta_{123} \right\} , \tag{11}
$$

with

$$
\Theta_{123} \equiv \theta(p_1^2+p_2^2-p_3^2) \; \theta(p_2^2+p_3^2-p_1^2) \; \theta(p_3^2+p_1^2-p_2^2) \tag{12}
$$

(cf. eq. (4.13) of [10]).

Of course, the same result (11) can be also obtained by putting $\nu_1=\nu_2=\nu_3=1$ in general results from [12,13] and using reduction formulae for the occurring F_4 functions, basically repeating the steps done in [10] for $I(4-2\varepsilon;1,1,1)$.

It is convenient to introduce the angles ϕ_i $(i=1,2,3)$ such that

$$
\cos\phi_1 = \frac{p_2^2+p_3^2-p_1^2}{2\sqrt{p_2^2 p_3^2}}, \quad \cos\phi_2 = \frac{p_3^2+p_1^2-p_2^2}{2\sqrt{p_3^2 p_1^2}}, \quad \cos\phi_3 = \frac{p_1^2+p_2^2-p_3^2}{2\sqrt{p_1^2 p_2^2}}, \tag{13}
$$

so that $\phi_1+\phi_2+\phi_3=\pi$, and the arguments of ${}_2F_1$ functions in eq. (11) are nothing but minus $\tan^2\phi_i$. Note that the angles θ_i from [9,10] are related to ϕ_i as $\theta_i = 2\phi_i$. By analogy with the two-point case (2), the angles ϕ_i can be understood as the angles of a triangle whose sides are $\sqrt{p_1^2}$, $\sqrt{p_2^2}$ and $\sqrt{p_3^2}$, whereas its area is $\frac{1}{4}\sqrt{\Delta(p_1^2,p_2^2,p_3^2)}$.

3. The crucial step in constructing the ε-expansion is the formula

$$
\sum_{j=0}^{\infty} \frac{(-2\varepsilon)^j}{j!} \, \mathrm{Ls}_{j+1}(2\phi) = - \frac{2^{1-2\varepsilon}\tan\phi}{(1-2\varepsilon)\sin^{2\varepsilon}\phi} \; {}_2F_1\left(\begin{array}{c} 1, 1/2 \\ 3/2-\varepsilon \end{array} \middle| -\tan^2\phi \right)
$$

$$
-2\pi \frac{\Gamma(1-2\varepsilon)}{\Gamma^2(1-\varepsilon)} \, \theta(-\cos\phi) \; . \tag{14}
$$

Using the definition (4), we see that the l.h.s. of eq. (14) is nothing but

$$
\sum_{j=0}^{\infty} \frac{(-2\varepsilon)^j}{j!} \, \mathrm{Ls}_{j+1}(2\phi) = -2^{1-2\varepsilon} \int_0^{\phi} \mathrm{d}\tau \; \sin^{-2\varepsilon}\tau \; . \tag{15}
$$

Evaluating this integral (treating the cases $0 \le \phi < \pi/2$ and $\pi/2 < \phi < \pi$ separately), we arrive at the r.h.s. of eq. (14), q.e.d.

In the special case $\phi = \pi/2$ we get (see in [4,6])

$$\sum_{j=0}^{\infty} \frac{(-2\varepsilon)^j}{j!} \, \mathrm{Ls}_{j+1}(\pi) = -\pi \, \frac{\Gamma(1-2\varepsilon)}{\Gamma^2(1-\varepsilon)} \, . \tag{16}$$

Now, using eq. (14) for all three ${}_2F_1$ functions occurring in eq. (11), identifying Θ_{123} (see eq. (12)) as

$$\Theta_{123} = 1 - \theta(-\cos\phi_1) - \theta(-\cos\phi_2) - \theta(-\cos\phi_3), \tag{17}$$

then using eq. (16) and shifting $j \to j+1$, we arrive at the following ε-expansion:

$$J(4-2\varepsilon;1,1,1) = 2\pi^{2-\varepsilon} \, \mathrm{i}^{1+2\varepsilon} \, \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \, \frac{\left[\Delta(p_1^2,p_2^2,p_3^2)\right]^{-1/2+\varepsilon}}{(p_1^2 p_2^2 p_3^2)^{\varepsilon}}$$
$$\times \sum_{j=0}^{\infty} \frac{(-2\varepsilon)^j}{(j+1)!} \left[\mathrm{Ls}_{j+2}\left(2\phi_1\right) + \mathrm{Ls}_{j+2}\left(2\phi_2\right) + \mathrm{Ls}_{j+2}\left(2\phi_3\right) - 2\mathrm{Ls}_{j+2}\left(\pi\right)\right] . \tag{18}$$

The shift of j was possible, since

$$\mathrm{Ls}_1\left(2\phi_1\right) + \mathrm{Ls}_1\left(2\phi_2\right) + \mathrm{Ls}_1\left(2\phi_3\right) - 2\mathrm{Ls}_1\left(\pi\right) = -2(\phi_1+\phi_2+\phi_3-\pi) = 0. \tag{19}$$

The ε-expansion of the two-loop vacuum integral $I(4-2\varepsilon;1,1,1)$ can be obtained via the magic connection, i.e. by substituting eq. (18) into eq. (23) of [9]:

$$I(4-2\varepsilon;1,1,1) = \pi^{4-2\varepsilon} \, \frac{\Gamma^2(1+\varepsilon)}{(1-\varepsilon)(1-2\varepsilon)} \, \left\{ \left[\Delta(m_1^2,m_2^2,m_3^2)\right]^{1/2-\varepsilon} \right.$$
$$\times \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{(j+1)!} \left[\mathrm{Ls}_{j+2}\left(2\phi_1\right) + \mathrm{Ls}_{j+2}\left(2\phi_2\right) + \mathrm{Ls}_{j+2}\left(2\phi_3\right) - 2\mathrm{Ls}_{j+2}\left(\pi\right)\right]$$
$$\left. - \frac{1}{2\varepsilon^2} \left[\frac{m_1^2+m_2^2-m_3^2}{(m_1^2 m_2^2)^{\varepsilon}} + \frac{m_2^2+m_3^2-m_1^2}{(m_2^2 m_3^2)^{\varepsilon}} + \frac{m_3^2+m_1^2-m_2^2}{(m_3^2 m_1^2)^{\varepsilon}} \right] \right\} . \tag{20}$$

When the masses are equal, $m_1 = m_2 = m_3 \equiv m$ (this also applies to the symmetric case $p_1^2 = p_2^2 = p_3^2 \equiv p^2$), the three angles ϕ_i are all equal to $\pi/3$, whereas $\Delta(m^2,m^2,m^2) = 3m^4$. Therefore, in this case the r.h.s. of eq. (20) becomes

$$\pi^{4-2\varepsilon} \, \frac{\Gamma^2(1+\varepsilon)\, m^{2-4\varepsilon}}{(1-\varepsilon)(1-2\varepsilon)} \left\{ 3^{1/2-\varepsilon} \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{(j+1)!} \left[3\mathrm{Ls}_{j+2}\left(\frac{2\pi}{3}\right) - 2\mathrm{Ls}_{j+2}\left(\pi\right) \right] - \frac{3}{2\varepsilon^2} \right\} . \tag{21}$$

For instance, in the contribution of order ε the transcendental constant $\mathrm{Ls}_3\left(2\pi/3\right)$ appears. This constant (and its connection with the inverse tangent integral value $\mathrm{Ti}_3(1/\sqrt{3})$) was discussed in detail in [9]. The fact that $\mathrm{Ls}_3\left(2\pi/3\right)$ occurs in certain two-loop on-shell integrals has been noticed in [14]. Moreover, in [15] it was observed

that the higher-j terms from (21) form a basis for certain on-shell integrals with a single mass parameter.

Note that the structure of eq. (18) is very similar to that of the two-point function with masses (2). For the two lowest orders (ε^0 and ε^1), we reproduce eqs. (9)–(10) from [9]. Useful representations for the ε^0 terms of both types of diagrams can also be found in [16]. We note that the ε-term of the one-loop massive three-point function was calculated in [7], whereas the massless case was considered in [17].

Moreover, in eq. (26) of [17] a one-fold integral representation for $J(4-2\varepsilon;1,1,1)$ is presented (for its generalization, see eq. (7) of [9]). Expanding the integrand in ε, we were able to confirm the ε-expansion (18) numerically.

4. We have shown that the compact structure of the coefficients of the ε-expansion of the two-point function (2), in terms of log-sine integrals, also takes place for the massless off-shell three-point function (18) and two-loop massive vacuum diagrams (20). It is likely that a further generalization of these results is possible, e.g. for the three-point function with different masses and some two-point integrals with two (and more) loops. In particular, numerical analysis of the coefficients of the expansion of certain two-point on-shell integrals and three-loop vacuum integrals [15] shows that in some cases the values of generalized log-sine integrals $\mathrm{Ls}_j^{(l)}$ (see eq. (7.14) of [6]) may be involved.

The fact that the generalization of $\mathrm{Ls}_2 = \mathrm{Cl}_2$ goes in the Ls_j direction, rather than in Cl_j direction (see eq. (5)), is very interesting. There is another example [18] (see also in [19]), the off-shell massless ladder three- and four-point diagrams with an arbitrary number of loops, when such a generalization went in the Cl_j direction. Just as an illustration, we can present the result for the L-loop function $\Phi^{(L)}(x,y)$ (for the definition, see eqs. (12) and (21) of [18]; $x \leftrightarrow p_1^2/p_3^2$, $y \leftrightarrow p_2^2/p_3^2$) in the case $y = x$, which is valid when $\Delta(p_1^2, p_2^2, p_3^2) < 0$:

$$\Phi^{(L)}(x,x) = \frac{(2L)!}{(L!)^2}\,\frac{1}{x\sin\theta}\,\mathrm{Cl}_{2L}\left(\theta\right), \qquad \theta = \arccos\left(1-\frac{1}{2x}\right). \tag{22}$$

When $x = 1$ ($p_i^2 = p^2$) this yields the $\mathrm{Cl}_{2L}\left(\pi/3\right)/\sqrt{3}$ structures. It could be also noted that the the two-loop non-planar (crossed) three-point diagram gives in this case the square of the one-loop function, $(\mathrm{Cl}_2\left(\theta\right))^2$ (cf. eq. (23) of [17]), leading to the structute $(\mathrm{Cl}_2\left(\pi/3\right))^2$ in the symmetric ($p_i^2 = p^2$) case. Recently, these constants have been also found in massive three-loop calculations [20, 21] (see also in [22]).

The representations (2), (18) and (20) are directly applicable to the case when the triangle function Δ given by eq. (3) is positive. When Δ is negative, we need to construct proper analytic continuation of Ls_j functions. For $j = 2$ this is simple, since $\mathrm{Ls}_2\left(\theta\right) = \mathrm{Cl}_2\left(\theta\right)$ and we can use the definition (5). Similarly one can deal with higher Cl_j functions. Let us consider the situation with analytic continuation of higher Ls_j functions. For $j = 3$, $\mathrm{Ls}_3\left(\theta\right)$ can be expressed in terms of the imaginary part of $\mathrm{Li}_3\left(1 - e^{\mathrm{i}\theta}\right)$, see in [6][1]. Using this fact, we can re-construct eq. (16) of the preprint version of [9], which gives the

[1] A factor $\frac{1}{2}$ is missing in front of $\mathrm{Ls}_3\left(\theta\right)$ in eq. (49) in p. 298 of [6], cf. eq. (6.56).

analytic continuation of the ε-term of eqs. (18) and (20). Then, the imaginary part of $\mathrm{Li}_4\left(1-e^{\mathrm{i}\theta}\right)$ is already a mixture of $\mathrm{Ls}_4(\theta)$ and $\mathrm{Cl}_4(\theta)$, see in [6][2], whereas its real part involves the generalized log-sine integral $\mathrm{Ls}_4^{(1)}(\theta)$ (see eq. (35) in p. 301 of [6]). Its value at $\theta = 2\pi/3$ also occurs in on-shell integrals considered in [15] as is shown to be connected with $V_{3,1}$ from [21]. The construction of analytic continuation of higher Ls_j functions is more cumbersome. In fact, it may require including the generalized polylogarithms (see, e.g., in ref. [23]).

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